To Isaac Dalby Esqr with the Authors respects

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THE

PRINCIPLES AND APPLICATION

OF

IMAGINARY QUANTITIES,

BOOK I;

TO WHICH ARE ADDED

SOME OBSERVATIONS

ON

Porisms;

BEING THE FIRST OF A SERIES OF

ORIGINAL TRACTS

ON VARIOUS PARTS OF THE MATHEMATICS.

By BENJAMIN GOMPERTZ, Esq.

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To MISS LOUSADA.

MADAM,

AS the mathematical studies are among the number of your scientific researches, and, in your pursuit, the more brilliant and nicer subjects have attracted your more particular attention, I have felt much flattered at the desire so frequently expressed by you for the publication of the following attempts which I have made to illustrate a branch of science by many considered difficult, being the first of a series of Mathematical Tracts which I am about to publish. The willingness with which you have agreed to have your name prefixed to this Essay, which probably but for your influence would never have been offered to fill the vacancies of a bookcase, encourages me to hope that it will not be thought wholly unworthy of public attention.

I am,

Madam,

with esteem,

your obedient and obliged humble Servant

BENJAMIN GOMPERTZ.

INTRODUCTION.

I. IMMEDIATELY after the invention of methods for the discovery of the roots of an equation above the first degree, algebraic expressions must have occurred, which seemed to require the extraction of the square root of a negative quantity; the impracticability of which could not have failed to throw much embarrassment in the way of the early promoters of algebraic knowledge. Ages of study on this subject have pointed out beauties where deformity appeared only to reign; and the reiterated attempts of analysts to unriddle the secrets of science, have been rewarded by the discovery that those very expressions termed impossible or imaginary quantities, which were terrific, sterile, and unproductive of utility, in appearance, were the most powerful instruments he could possess. But, notwithstanding the acquirement of the practice of these instruments of analysis, the method of their operation is far from being universally known; and the truths of their results are, by no means, generally acknowledged to be legitimately obtained, and would, by many, be wholly rejected in the absence of other demonstrations.

A common inference naturally to be drawn from the appearance of an expression of this sort in the analysis of a problem, is, that the problem is itself impossible; as if it were required to find the number, such that the excess of its double above its square shall be equal to 2. Putting x for this number, we shall, according to the conditions of the problem, have $2x-x^2=2$; and, by the common method of resolving quadratic equations, we get $x=1\pm\sqrt{-1}$. Here the impossible quantity makes its appearance; and here, indeed, the question is impossible.

II. If the question were to find that number whose double shall exceed its square by the greatest quantity; if this excess be put $=m$, we shall have $2x-x^2=m$, and consequently $x=1\pm\sqrt{1-m}$; but if m be greater than 1, $1-m$ will be negative, and consequently its square

root impossible. The greatest value that m can be, to avoid the impossibility, is therefore 1; and then x is 1; and this answers the question. This example already shows that a regard to the impossible quantity is serviceable in analysis: here the advantage, however, is not reaped from the introduction of the expression, but from the manner of avoiding it, the knowledge of which is attained from a consideration of the cause of its production; and it is to this cause that the Reader's attention will be drawn in my endeavours to explain the propriety of its use.

III. It will, perhaps, be useful, even in the present stage of this discourse, to observe that, as in different speculations there are different sources from whence impossibility may arise, so do we properly find different expressions of impossibility: thus, if x had been required so that $4x-x^2$ should be equal to m, we should obtain $x=2\pm\sqrt{4-m}$; but had it been required to make $2x-x^2=m$, we should have $x=1\pm\sqrt{1-m}$. Now, were $m=10$, we should have in the one case $x=2\pm\sqrt{-6}$, and in the other $x=1\pm\sqrt{-9}$; and we see that m will admit of a greater value, without exceeding the limits of possibility, in the first case than in the second; for m can be 4 in the first case, but cannot be greater than 1 in the second.

IV. But impossible quantities are well known to occur also in the resolution of possible questions; for instance, in the resolution of the cubic equation $x^3-63x=162$, having three possible answers for x, by means of Cardan's Rule; and still, from these very impossible quantities, by an analysis properly conducted, we have the means of discovering the answers.

V. In conducting the operation by the algebraic expressions termed impossible or imaginary quantities, the same rules are observed as in other parts of algebra; thus, if $\sqrt[2]{-a}$ were to be multiplied by c, its product would be considered $c.\sqrt[2]{-a}$; if $\sqrt[2]{-a}$ were to be multiplied by $\sqrt[2]{-b}$, the product would be considered $\sqrt[2]{-a\times-b}$ or $\sqrt[2]{ab}$. But before I proceed, I think it necessary to inform the Reader that there are many persons of respectable mathematical acquirements, and of

sound judgment in various branches of human research, who consider the operations of these quantities absurd; and the profound mathematicians Newton, Euler, &c. in lieu of receiving the admiration due to their penetration and ingenuity in this part of science, have been hastily considered, by some, to be bad reasoners; or, at least, to have vitiated mathematical arguments.

VI. The foundation of the rules of operation appears to have been originally derived from the laws of continuity; thus, if we have $x=\sqrt{a}$, $y=\sqrt{b}$, we shall have $xy=\sqrt{ab}$; and if we were to diminish a continually, this would be always true till a became equal 0; and if a were to be conceived still diminished until it became $=-c$, the thing might be imagined still to be true; and, substituting $-c$ for a, we should have $\sqrt{a}\times\sqrt{b}=\sqrt{-c}\times\sqrt{b}=\sqrt{-c\times b}=\sqrt{-bc}$; and, in a similar manner, if b be supposed to be diminished till it becomes equal to $-d$, imagining the expression $\sqrt{b}$ still to be significant, and the truth of the original operation not be destroyed, we shall have $\sqrt{a}.\sqrt{b}=\sqrt{-c}.\sqrt{-d}=\sqrt{-c\times-d}=\sqrt{cd}$. The objection raised against this operation is, that as the square roots of negative numbers do not exist in arithmetic, reasoning in this manner must be inconsistent. How far the objection here named is just I shall not, at this part of my Essay, consider; though I think it proper to observe, that some followers of this branch of analysis, among whom is the celebrated Emerson, very erroneously, in my opinion, consider $\sqrt{-c}.\sqrt{-d}$ to be $=\sqrt{-dc}$. The reason they give for this assertion is, that if it were $=\sqrt{dc}$, a real product would be raised from impossible factors. This argument appears to me quite contrary to the spirit of the method; for, without at this moment being anxious to declare on the justness of the method of operating with expressions termed imaginary, I consider it proper to remind the operator that the idea of $x=\sqrt{-c}$ and $y=\sqrt{-d}$ is derived from the supposition of $x^2=-c$ and $y^2=-d$; that is, that $-c=\sqrt{-c}\times\sqrt{-c}$ and $-d=\sqrt{-d}.\sqrt{-d}$; that is, that a real negative product is derived from imaginary factors; and should it be objected that a negative producted

is not a real product, still considering $x^2y^2=$ some product cd, it will likewise be equal to $\overline{-c}\times\overline{-d}$, which is the same thing in an analytical point of view, allowing the law of continuity above cited to be admissible. The beautiful application of the imaginary algebra would be lost to any one maintaining the idea that $\sqrt{-c}\,.\,\sqrt{-d}$ were $=\sqrt{-cd}$, for such an idea would lead to error. There is another thing with which some mathematicians have been puzzled: they say, that though it appears that $\sqrt{-c}\times\sqrt{-c}$ is $=-c$, it equally appears that $\sqrt{-c}\times\sqrt{-c}$ $=\sqrt{-c\times-c}=\sqrt{c^2}=c$; but the fact is, that $\sqrt{c^2}$ is neither equal to $+c$ nor to $-c$; that is, it is not identical with either, but that it is an expression having a double value, of which the one is $+c$ and the other $-c$; and that in consequence, so far from the thing being an absurdity, it is a proof of the excellence of the symbol $\sqrt{}$. This symbol contains the two signs $+$ and $-$; and the difficulty does not particularly belong to the imaginary quantity. The product $\sqrt{c}\,.\sqrt{c}$ ought to contain two signs $+$ and $-$. If the two symbols (both expressed by $\surd$) are to have the same sign, in which case $\sqrt{c}\,.\,\sqrt{c}$ will be the same with $\sqrt{\overline{c}|^2}$, the value will be $+c$; but if they have different signs it will be $-c$; and in this case it is not the same as $\sqrt{\overline{c}|^2}$. And, in like manner, when the symbol $\surd$ is to be taken with the same sign in the two factors of $\sqrt{-c}$, $\sqrt{-c}$, then will $\sqrt{-c}\,.\,\sqrt{-c}$ be the same with $\sqrt{\overline{-c}|^2}=-c$; but if they are to have different signs, it will be $=+c$. It is of great moment, in various parts of mathematics, to be acquainted with the distinction between the equality of expressions and the different interpretations of which they will admit.

I have, in the above observations, pointed out what I conceive to have been the original steps which led to the introduction of imaginary quantities into algebra, and have shewn what an assumed law of continuity ought to lead to in the operation. But it is not on an assumed law that I mean to rest; it is my object, in the First Book of this Treatise, to explain the principles of this method by a mode, I believe, quite different to any hitherto pursued; the whole of this book being meant to illustrate the following position of the First Book.

BOOK I.

ON IMAGINARY QUANTITIES.

POSITION.

That wherever the operation by imaginary expressions can be used, the propriety may be explained from the capability of one arbitrary quantity or more being introduced into the expressions which are imaginary previously to the said arbitrary quantity or quantities being introduced; so as to render them real, without altering the truth they are meant to express; and that, in consequence, the operation will proceed on real quantity; the introduced arbitrary quantity or quantities necessary to render the first steps of the reasoning arguments on real quantity, vanishing at the conclusion; and from whence it will follow that the non-introduction of such can produce nothing wrong.

1. Though I do not call to mind ever having met with this mode of proceeding, I am well aware of the hazard and even injustice in the present advanced state of mathematics, to affirm any particular view of a subject to be new: and, therefore, leaving every one to make his claim, I shall commence by considering the factors of the expression x^2+ax+b, which are discovered, according to the common method, by feigning $x^2+ax+b=0$; whence it is found that the factors are $x+\frac{a}{2}+\sqrt{\frac{a^2}{4}-b}$, and $x+\frac{a}{2}-\sqrt{\frac{a^2}{4}-b}$; and that, in consequence, when b is greater than $\frac{a^2}{4}$, these become imaginary; and that, therefore, any analysis which proceeds by these factors will, in this case, proceed by imaginary quan-

tities; and that this may be avoided, let us assume $x^2+ax+b=\overline{p+\sqrt{\frac{a^2}{4}-b+\varrho}}\times\overline{p-\sqrt{\frac{a^2}{4}-b+\varrho}}$, or its equal $p^2-\frac{a^2}{4}+b-\varrho$; whence we find $p=\sqrt{x^2+ax+b+\frac{a^2}{4}-b+\varrho}=\sqrt{\overline{x+\frac{a}{2}}\,^2+\varrho}$; and, consequently, we find $x^2+ax+b=\left(\sqrt{\overline{x+\frac{a}{2}}\,^2+\varrho}+\sqrt{\frac{a^2}{4}-b+\varrho}\right)\times$ $\left(\sqrt{\overline{x+\frac{a}{2}}\,^2+\varrho}-\sqrt{\frac{a^2}{4}-b+\varrho}\right)$; ϱ being an arbitary quantity, and consequently may always be chosen, so that the factors shall be real quantities. It is evident that if we take $\varrho=0$, the factors will be as first determined, $x+\frac{a}{2}+\sqrt{\frac{a^2}{4}-b}$, and $x+\frac{a}{2}-\sqrt{\frac{a^2}{4}-b}$.

Taking $a=0$, we shall have, for the factors of x^2+b, $\sqrt{x^2+\varrho}+\sqrt{\varrho-b}$; and $\sqrt{x^2+\varrho}-\sqrt{\varrho-b}$, which if ϱ were taken $=0$, we observe would be the imaginary factors $x+\sqrt{-b}$, and $x-\sqrt{-b}$, commonly used, and which it was our object to avoid.

If we had the expression x^3+gx^2+hx+k to reduce into factors, only having one possible binomial factor suppose, g, h and k being real quantities, the expression may be written $\overline{x+c}\times\overline{x^2+ax+b}$; a, b and c being real quantities, and consequently, from the above, this may be written $\overline{x+c}\times\left(\sqrt{\overline{x+\frac{a}{2}}\,^2+\varrho}+\sqrt{\frac{a^2}{4}-b+\varrho}\right)\times\left(\sqrt{\overline{x+\frac{a}{2}}\,^2+\varrho}-\sqrt{\frac{a^2}{4}-b+\varrho}\right)$, ϱ being an arbitrary quantity: thus, if the expression were x^3-a^3, or its equal $\overline{x-a}\times\overline{x^2+ax+a^2}$, it would likewise be $(x-a)\left(\sqrt{\overline{x+\frac{a}{2}}\,^2+\varrho}+\sqrt{\varrho-\frac{3}{4}a^2}\right)$ $\times\left(\sqrt{\overline{x+\frac{a}{2}}\,^2+\varrho}-\sqrt{\varrho-\frac{3}{4}a^2}\right)$. If ϱ be $=0$, it will become $\overline{x-a}\left(x+\frac{a}{2}+a\right.$ $\left.\cdot\sqrt{-\frac{3}{4}}\right)\times\left(x+\frac{a}{2}-a\sqrt{-\frac{3}{4}}\right)$.

2. In Euler's Algebra we find the impossible arithmetical quantity applied, with advantage, to the determination of x and y, such, that the formula $ax^2+bxy+cy^2$ shall have factors: this, when $a=1$ and $b=0$, is x^2+cy^2; and, making no objection to the impossible quantities, the factors are $x+y\sqrt{-c}$ and $x-y\sqrt{-c}$; these are then assumed respectively equal to $\overline{p+q\sqrt{-c}}\times\overline{r+s\sqrt{-c}}$ and $\overline{p-q\sqrt{-c}}\times\overline{r-s\sqrt{-c}}$, and we thence get $\overline{p^2+cq^2}\times\overline{r^2+cs^2}=x^2+cy^2$, and x is found $=pr-cqs$ and $y=ps+qr$, or $x=pr+cqs$, and $y=ps-qr$. The whole of this takes for granted the propriety of the rules practised with impossible quantities; but, according to our method, we will consider the formula $x^2+cy^2=(\sqrt{\varrho+x^2}-\sqrt{\varrho-cy^2})\times(\sqrt{\varrho+x^2}+\sqrt{\varrho-cy^2})$, a truth evident from putting cy^2 for b in the above resolution of x^2+b into real factors. Now put the factor $\sqrt{\varrho+x^2}-\sqrt{\varrho-cy^2}=(\sqrt{\varrho'+p^2}-\sqrt{\varrho'-cq^2})\times(\sqrt{\varrho''+r^2}-\sqrt{\varrho''-cs^2})$, and the factor $\sqrt{\varrho+x^2}+\sqrt{\varrho-cy^2}=(\sqrt{\varrho'+p^2}+\sqrt{\varrho'-cq^2})\times(\sqrt{\varrho''+r^2}+\sqrt{\varrho''-cs^2})$ steps which would be identical with Euler's, were ϱ, ϱ', and $\varrho''=0$; and consequently by multiplying we get $x^2+cy^2=\overline{p^2+cq^2}\times\overline{r^2+cs^2}$, as above, the three introduced arbitrary quantities ϱ, ϱ', ϱ'' vanishing of themselves. Now, to find x and y: by addition and subtraction, we get $2\sqrt{\varrho+x^2}=2\sqrt{\varrho'+p^2}.\sqrt{\varrho''+r^2}+2\sqrt{\varrho'-cq^2}.\sqrt{\varrho''-cs^2}$; and $2\sqrt{\varrho-cy^2}=2\sqrt{\varrho'+p^2}.\sqrt{\varrho''-cs^2}+2\sqrt{\varrho'-cq^2}.\sqrt{\varrho''+r^2}$; hence, dividing by 2, and squaring and putting R for $2\varrho'\varrho''+\varrho'r^2+\varrho''p^2-\varrho'cs^2-\varrho''cq^2+2\sqrt{\varrho'+p^2}.\sqrt{\varrho''+r^2}.\sqrt{\varrho'-cq^2}.\sqrt{\varrho''-cs^2}$ for the sake of brevity, we shall find $\varrho+x^2=p^2r^2+c^2s^2q^2+\mathrm{R}$, and $\varrho-cy^2=\mathrm{R}-cs^2p^2-cq^2r^2$; that is, $x^2=p^2r^2+c^2s^2q^2+\mathrm{R}-\varrho$, and $cy^2=cs^2p^2+cq^2r^2-\mathrm{R}+\varrho$. Now it is evident that as ϱ, ϱ', and ϱ'' are all arbitrary, we may take them so that $\mathrm{R}-\varrho=\pm 2cspr$, without the introduction of any imaginary quantity; and we shall, in consequence, have, as above, $x=pr\pm cqs$ and $y=ps\mp qr$. And thus I have shown how, by the introduction of arbitrary quantities, we are enabled to imitate Euler's steps, which proceed by imaginary quantities, and to obtain his result; and we have only to suppose (ϱ) one of these arbitrary quantities $=\mathrm{R}\pm 2cspr$, without being at any other trouble but the bare supposition, no subsequent calculation being necessary with the other arbitrary quantities ϱ', ϱ'', as they will of themselves vanish out of the equation; and we, in consequence, see that, had

we not introduced them, we should not have done any thing wrong; and, in which case, the analysis would have been the same as Euler's. I may remark, by the bye, that if $r=pm$, and $s=qm$, we shall have $x=m\,.\,(p^2c\mp q^2)$, and $y=2mp$ or 0, according as the upper or under sign is used; and, consequently, we find (with Euler, p. 156, Algebra, English translation) that if $x=m.(p^2-cq^2)$ and $y=2mpq$, x^2+cy^2 is a square, and $=m^2\,.\,\overline{p^2+cq^2}|^2$.

3. Let it now be required to resolve the equation $x^3-ax=b$, a and b being given quantities.

For this purpose, take $x=ny+\frac{m}{y}$; $\therefore\ x^3=n^3y^3+3n^2my+\frac{3nm^2}{y}+\frac{m^3}{y^3}=n^3y^3+\frac{m^3}{y^3}+3nm\left(ny+\frac{m}{y}\right)=n^3y^3+\frac{m^3}{y^3}+3mnx$; consequently the equation $x^3-ax=b$ becomes $n^3y^3+\frac{m^3}{y^3}=b+\varrho x$, putting ϱ for $a-3nm$; and, squaring, we get $n^6y^6+2n^3m^3+\frac{m^6}{y^6}=\overline{b+\varrho x}|^2$. Take away $4n^3m^3$ from both sides, and extract the square root, and we shall have $n^3y^3-\frac{m^3}{y^3}=\sqrt[3]{\overline{b+\varrho x}|^2-4n^3m^3}$; this, compared with the equation $n^3y^3+\frac{m^3}{y^3}=b+\varrho x$, gives $ny=\sqrt[3]{\frac{b+\varrho x}{2}+\sqrt[3]{\frac{\overline{b+\varrho x}|^2}{2}-n^3m^3}}$; and $\frac{m}{y}=\sqrt[3]{\frac{b+\varrho x}{2}-\sqrt[2]{\frac{\overline{b+\varrho x}|^2}{2}-n^3m^3}}$; and, consequently, because $nm=\frac{a-\varrho}{3}$, we get x or $ny+\frac{m}{y}=\sqrt[3]{\frac{b+\varrho x}{2}+\sqrt[2]{\frac{\overline{b+\varrho x}|^2}{2}+\overline{\frac{a-\varrho}{3}}|^3}}+\sqrt[3]{\left(\frac{b+\rho x}{2}-\sqrt[2]{\left(\overline{\frac{b+\varrho x}{2}}|^2-\overline{\frac{a-\varrho}{3}}|^3\right)}\right)}$, ϱ being an arbitrary letter.

If we take $\varrho=0$, we shall get the common formula $x=\sqrt[3]{\frac{b}{2}+\sqrt[2]{\frac{b^2}{4}-\frac{a^3}{27}}}+\sqrt[3]{\frac{b}{2}-\sqrt[2]{\frac{b^2}{4}-\frac{a^3}{27}}}$, which will become imaginary when $\frac{a^3}{27}$ is greater than

$\frac{b^2}{4}$; and it is easy to see that ϱ may be so taken that the imaginary quantities may be avoided; but as x would be then contained within the vincula, some further expedient would be necessary. If, for instance, the equation had been $x^3-63x=162$, we should have $x=\sqrt[3]{81+\frac{1}{2}\varrho x+\sqrt[2]{\overline{81+\frac{1}{2}\varrho x}^2-\overline{21-\frac{1}{3}\varrho}^3}}+\sqrt[3]{81+\frac{1}{2}\varrho x-\sqrt{\overline{81+\frac{1}{2}\varrho x}^2-\overline{21-\frac{1}{3}\varrho}^3}}$; here, when $\varrho=0$, we get $x=\sqrt[3]{81+\sqrt[2]{-2700}}+\sqrt[3]{81-\sqrt[2]{-2700}}$: but when ϱ is not taken $=0$, it will not, I think, be improper to show that if for the x within the vincula we put one of its values, 9 for instance, we get $x=\sqrt[3]{81+\frac{9}{2}\varrho+\sqrt[2]{\overline{81+\frac{9}{2}\varrho}^2-\overline{21-\frac{1}{3}\varrho}^3}}+\sqrt[3]{\left(81+\frac{9}{2}\cdot\varrho-\sqrt[2]{\overline{81+\frac{9}{2}\cdot\varrho}^2-\overline{21-\frac{1}{3}\varrho}^3}\right)}$, and this will be so at least when real, without any regard to the value of ϱ, as ϱ and x are quite independent of each other. Thus, take $\varrho=63$; then is $x=\sqrt[3]{81+9\times\frac{63}{2}+\sqrt[2]{\overline{81+\frac{9}{2}63}^2-0}}+\sqrt[3]{81+9\times\frac{63}{2}-\sqrt[2]{\overline{81+\frac{9}{2}63}^2-0}}=\sqrt[3]{162+567}+0=\sqrt[3]{729}=9$. If ϱ be taken $=6$, we shall equally have $x=9$; for the theorem will become $x=\sqrt[3]{108+\sqrt[2]{4805}}+\sqrt[3]{108-\sqrt[2]{4805}}=5.61804$, nearly, $+3.38196$ nearly $=9$; and so we might proceed with any other value of ϱ which would not lead to imaginary quantities.

The Reader ought to be reminded, that our theorem, though it appears of service to explain the office of the imaginary quantity in Cardan's Rule, still, without further expedient, it does not seem of service for the discovery of the value of x; as, in the above examples, we supposed the value of x previously known, to enable us to subject our formula to calculation. But still we see that if we extract the roots expressed in the formula, $x=\sqrt[3]{\frac{b+\varrho x}{2}+\sqrt[2]{\overline{\frac{b+\varrho x}{2}}^2-\overline{\frac{a-\varrho}{3}}^3}}+\sqrt[3]{\left(\frac{b+\varrho x}{2}-\right.}$

$\left.\sqrt[2]{\left(\frac{b+\varrho x}{2}\right)^2-\left(\frac{a-\varrho}{3}\right)^3}\,\right)$ by means of infinite series, all terms introduced, which are multiplied by ϱ or its powers, ought to vanish of themselves. At present, I shall only consider the case in which $\frac{4a^3}{27b^2}-1$ is a proper fraction; and first show the Reader who may be unacquainted with it, a method to be used in Cardan's failing cases to get rid of the imaginary quantities, by those who do not object to its operation. Put $\frac{4a^3}{27b^2}-1=k$; then will $\sqrt[3]{\frac{b}{2}+\sqrt[2]{\frac{b^2}{4}-\frac{a^3}{27}}}$ be $=\sqrt[3]{\frac{b}{2}}\cdot\sqrt[3]{1+\sqrt{-k}}=$ (by the binomial theorem) $\sqrt[3]{\frac{b}{2}}\times\left(1+\frac{1}{3}\sqrt{-k}-\frac{2}{3.6}\left(\sqrt{-k}\right)^2+\frac{2.5}{3.6.9}\left(\sqrt{-k}\right)^3,\ \&c\right)$, and in the same way $\sqrt[3]{\frac{b}{2}-\sqrt[2]{\frac{b^2}{4}-\frac{a^3}{27}}}$ is equal to $\sqrt[3]{\frac{b}{2}}\times\left(1-\frac{1}{3}\sqrt{-k}-\frac{2}{3.6}\left(\sqrt{-k}\right)^2-\frac{2.5}{3.6.9}\left(\sqrt{-k}\right)^3,\ \&c.\right)$; consequently, x being the sum of these two, and considering what has already been observed to be regarded by the operators with imaginary quantities, namely, that $\left(\sqrt{-k}\right)^2=-k$; and therefore $\left(\sqrt{-k}\right)^4=k^2$, $\left(\sqrt{-k}\right)^6=-k^3$, &c. we have $x=\sqrt[3]{4b}\times\left(1+\frac{2}{3.6}k-\frac{2.5.8}{3.6.9.12}\cdot k^2,\ \&c.\right)$ for one root of the proposed equation.

Now, according to the proposed object, I am to show that the analysis by the imaginary quantity can be dispensed with, in using my theorem for the same investigation; that is, that if $x^3-ax=b$, $x=\sqrt[3]{\frac{b+\varrho x}{2}+\sqrt[2]{\left(\frac{b+\varrho x}{2}\right)^2-\left(\frac{a-\varrho}{3}\right)^3}}+\sqrt[3]{\frac{b+\varrho x}{2}-\sqrt[2]{\left(\frac{b+\varrho x}{2}\right)^2-\left(\frac{a-\varrho}{3}\right)^3}}$. Put, as before, $\frac{4a^3}{27b^2}-1=k$; and also put, for the sake of brevity, $R=\frac{\varrho^3-3a\varrho^2+3a^2\varrho}{27\times\frac{1}{4}b^2}$

$-\frac{2b\varrho x-\varrho^2x^2}{b^2}$, and we shall have $x=\sqrt[3]{\frac{b}{2}}\times\left(\sqrt[3]{1+\frac{\varrho}{b}x+\sqrt[2]{R-k}}+\right.$ $\left.\sqrt[3]{1+\frac{\varrho}{b}x-\sqrt{R-k}}\right)$; and if we develope this by the binomial theorem, we shall have it $=\sqrt[3]{4b}\times\left(\overline{1+\frac{\varrho}{b}x}\right|^{\frac{1}{3}}-\frac{2}{3.6}\frac{R-k}{\overline{1+\frac{\varrho}{b}x}\big|^{\frac{5}{3}}}-\frac{2.5.8}{3.6.9.12}\frac{\overline{R-k}\big|^2}{\overline{1+\frac{\varrho}{b}x}\big|^{\frac{8}{3}}}$, &c.$\Big)$ $=$ $\sqrt[3]{4b}\times\left(1+\frac{2}{3.6}k-\frac{2.5.8}{3.6.9.12}k^2,\&c.+R'\right)$, R' being put for all the terms of which every part is concerned with some power of ϱ; but, as ϱ is independent of x, all terms which introduce it must vanish of themselves, that is, must destroy each other, and therefore R' represents 0; and therefore $x=\sqrt[3]{4b}\times\left(1+\frac{2}{3.6}k-\frac{2.5.8}{3.6.9.12}k^2,\ \&c.\right)$, the same as before.

4. Imaginary quantities have been used by Waring and Simpson (see Waring's Meditationes Algebraicæ, 3d Edition, Preface, page xxviii, and in the Work page 157) in the solution of the following general problem:

"A quantity being given which expresses a series $(a+b+c+d+e+f+g+h+k+$ &c.) of simple terms, proceeding according to the dimensions 0, 1, 2, 3, &c. of any letter x, to find a quantity equal to the sum of the alternate terms of the series $(a+c+e+g+k+$ &c.); and likewise quantities respectively equal to the sums of the terms of the proposed series, standing at the 2d, 3d, or, lastly, the nth interval from each other."

The impossible quantity being required in their analysis of the problem, when two or more of the terms of the original series are to be continually omitted, I shall consider the case only when there are two of the terms continually omitted, as the method of proceeding when there are more of them left out will be easily perceived from this:

Thus, let the proposition be, Given $S=a+bx+cx^2+dx^3+ex^4+fx^5+gx^6+$, &c. to find the sum of the series $a+dx^3+gx^6+$, &c. Then according to

the method of imaginary quantities above alluded to, put αx, βx, γx, in the room of x, and let S become, by that substitution, A, B, C, and consequently we have

$$\left.\begin{array}{l} A=a+b\,.\,\alpha x+c\alpha^2x^2+d\alpha^3x^3+\ \&c. \\ B=a+b\beta\,.\,x+c\beta^2\,.\,x+d\beta^3x^3+\ \&c. \\ C=a+b\gamma x\ \ +c\gamma^2x\ \ +d\gamma^3x^3+\ \&c. \end{array}\right\}\text{ and one third of the sum of}$$

these three will give $\frac{A+B+C}{3}=a+b\,.\,\frac{\alpha+\beta+\gamma}{3}x+c\,.\,\frac{\alpha^2+\beta^2+\gamma^2}{3}x^2+d\,.\,\frac{\alpha^3+\beta^3+\gamma^3}{3}$ $.\,x^3+$, &c.; consequently if $\alpha+\beta+\gamma=0$, $\alpha^2+\beta^2+\gamma^2=0$, $\alpha^3+\beta^3+\gamma^3=1$, we shall have, for the determination of α, β, γ, the three equations $\alpha^3-1=0$, $\beta^3-1=0$, $\gamma^3-1=0$; this will easily be seen by extermination, or from the theory of equations admitting of the doctrine of impossible quantities; and it is easy to see that α, β, γ, cannot be the same with each other, but are, according to the operations of *Imaginaries*, the different roots of the equation $z^3-1=0$. And it will likewise follow, that whatever number ν may represent that $\alpha^{\nu+3}+\beta^{\nu+3}+\gamma^{\nu+3}=\alpha^\nu+\beta^\nu+\gamma^\nu$; α^3, β^3, γ^3 being each $=1$; and therefore, because in the new series the coefficients of x and x^2 are $=0$, that is, if $\nu=1$ or 2, $\alpha^\nu+\beta^\nu+\gamma^\nu$ will be equal to 0; so will it follow that the coefficients of all powers of x, except those which are multiples of 3, will vanish; and it is also plain that all power of x which are multiples of 3 will be unity; and, consequently, that $\frac{A+B+C}{3}=a+dx^3+gx^6+$ &c.

This method of Waring, which has much elegance, considers of the three letters α, β, γ, one to be $=1$, one to be $=-\frac{1}{2}+\sqrt{-\frac{3}{4}}$, and the other to be $-\frac{1}{2}-\sqrt{-\frac{3}{4}}$. But if we take $\alpha=1$, $\beta=-\frac{1}{2}+\sqrt{\varrho-\frac{3}{4}}$, and $\gamma=-\frac{1}{2}-\sqrt{\varrho-\frac{3}{4}}$, and ϱ not less than $\frac{3}{4}$, they will not be imaginary; and then, to use similar steps, I observe that $\alpha+\beta+\gamma=0$, $\alpha^2+\beta^2+\gamma^2=2\varrho$,

$\alpha^3=1, \beta^3=1-\frac{3}{2}\varrho+\varrho\sqrt{\varrho-\frac{3}{4}}$, and $\gamma^3=1-\frac{3}{2}\varrho-\varrho\sqrt{\varrho-\frac{3}{4}}$; therefore $\alpha^3+\beta^3+\gamma^3=3-3\varrho$. Also, $\alpha^{\nu+3}+\beta^{\nu+3}+\gamma^{\nu+3}=\alpha^\nu+\beta^\nu+\gamma^\nu+\beta^\nu\cdot\left(\varrho\sqrt{\varrho-\frac{3}{4}}-\frac{3}{2}\varrho\right)-\gamma^\nu\cdot\left(\varrho\sqrt{\varrho-\frac{3}{4}}-\frac{3}{2}\varrho\right)$; or, generally, if we put $m=\varrho\sqrt{\varrho-\frac{3}{4}}-\frac{3}{2}\varrho$, and $n=-\varrho\sqrt{\varrho-\frac{3}{4}}-\frac{3}{2}\varrho$; since we shall have $\beta^3=1+m$, and $\gamma^3=1-n$, we have $\beta^{\nu+3}=\overline{1+m}.\beta^\nu$, and $\gamma^{\nu+3}=\overline{1-n}.\gamma^\nu$; also $\beta^{\nu+3t}=\overline{1+m}|^t.\beta^\nu=\beta^\nu+\beta^\nu\left(\overline{1+m}|^t-1\right)$, and $\gamma^{\nu+3t}=\gamma^\nu+\gamma^\nu\times\left(\overline{1-n}|^t-1\right)$, t being a whole number; therefore $\alpha^{\nu+3t}+\beta^{\nu+3t}+\gamma^{\nu+3t}=\alpha^\nu+\beta^\nu+\gamma^\nu+\beta^\nu\left(\overline{1+m}|^t-1\right)+\gamma^\nu\left(\overline{1-n}|^t-1\right)$, as α is $=1$.

Take $\nu=1$; therefore $\alpha^{1+3t}+\beta^{1+3t}+\gamma^{1+3t}=\beta.\left(\overline{1+m}|^t-1\right)+\gamma.\left(\overline{1-n}|^t-1\right)$, because $\alpha+\beta+\gamma=0$. Take $\nu=2$; therefore $\alpha^{2+3t}+\beta^{2+3t}+\gamma^{2+3t}=2\varrho+\beta^2\left(\overline{1+m}|^t-1\right)+\nu\left(\overline{1-n}|^t-1\right)$, because $\alpha^2+\beta^2+\gamma^2=2\varrho$. Take $\nu=3$; therefore $\alpha^{3+3t}+\beta^{3+3t}+\gamma^{3+3t}=3-3\varrho+\beta^3\left(\overline{1+m}|^t-1\right)+\gamma^3((1-n)^t-1)$, because $\alpha^3+\beta^3+\gamma^3=3-3\varrho$. Therefore $\frac{A+B+C}{3}=a+dx^3+gx^6+$ &c. $+R$, R standing for a series of quantities, every term of which is concerned with (that is, multiplied into) some positive power of ϱ; for $\left(\overline{1-m}|^t-1\right)$ has every term concerned with a positive power of m, and therefore with a positive power of ϱ. And the like of $(1-n)^t-1$; therefore we have $\frac{A+B+C}{3}-R=a+dx^3+gx^6$, &c. If ϱ had all the time been taken equal to 0, many of the above steps would not have appeared, and the analysis would have been the imaginary analysis above delivered, but not resorting to that case, as ϱ is independent of the value of the above series;

consequently all terms concerned with ϱ must destroy each other; and therefore, after proper reduction, we shall have the same value for the sum, whatever ϱ be taken.

Thus, we know that $\frac{1}{\sqrt{1-x}}=1+\frac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+\frac{1.3.5.7}{2.4.6.8}x^4+$ &c.; and if we wish to have the expression, whose developement is $1+\frac{1.3.5}{2.4.6}x^3+\frac{1.3.5.7.11.13}{2.4.6.8.10.12}x^6+$ &c., we have it, from the theorem above,

$$=\frac{1}{3\sqrt{1-x}}+\frac{1}{3\sqrt{1-x\times\left(-\frac{1}{2}+\sqrt{\varrho-\frac{3}{4}}\right)}}+\frac{1}{3\sqrt{1+x\left(\frac{1}{2}+\sqrt{\varrho-\frac{3}{4}}\right)}}-\mathrm{R}=$$

$$\frac{1}{3\sqrt{1-x}}+\frac{\sqrt{1+x\left(\frac{1}{2}+\sqrt{\varrho-\frac{3}{4}}\right)}+\sqrt{1-x\left(-\frac{1}{2}+\sqrt{\varrho-\frac{3}{4}}\right)}}{3\sqrt{\overline{1+\frac{1}{2}x}^2-\varrho-\frac{3}{4}.x^2}}-\mathrm{R}=$$ (by squaring and indicating the extraction of the square root of the denominator of the second fraction for the sake of reduction) $\frac{1}{3\sqrt{1-x}}+$

$$\frac{\sqrt{2+x+2\sqrt{1+x+x^2-\varrho x^2}}}{3\sqrt{1+x+x^2-\varrho x^2}}-\mathrm{R};$$ and consequently if this be developed, and we remember that in that case all terms in which ϱ is concerned must destroy each other, the expression will become $=\frac{1}{3\sqrt{1-x}}$

$$+\frac{\sqrt{2+x+2\sqrt{1+x+x^2}}}{3\sqrt{1+x+x^2}}.$$

It is very far from my wish to crowd this Essay with examples; but as the solution of the following will require an artifice not yet shown, I consider it proper to offer it for the Reader's attention. We know that $\overline{1-x}^{\frac{1}{3}}=1-\frac{1}{3}x-\frac{2}{3.6}x^2-\frac{2.5}{3.6.9}x^3$, &c.; consequently we have, from the above

method, $1-\frac{2.5x^3}{3.6.9}-\frac{2.5.8.11.14}{3.6.9.12.15.18}x^6-\&c.=\frac{\overline{1-x}|^{\frac{1}{3}}+w}{3}-R$, w being put for $\overline{1+\frac{1}{2}x-x\sqrt{\varrho-\frac{3}{4}}}|^{\frac{1}{3}}+\overline{1+\frac{1}{2}x+x\sqrt{\varrho-\frac{3}{4}}}|^{\frac{1}{3}}$; and therefore $w^3=2+x+3.\overline{\overline{1+\frac{1}{2}x}|^2-x^2.\varrho-\frac{3}{4}}|^{\frac{1}{3}}w=2+x+3.\overline{1+x+x^2-\varrho x^2}|^{\frac{1}{3}}w$. Now let $u^3=2+x+3.\overline{1+x+x^2}|^{\frac{1}{3}}.u$; and as the equation immediately preceding, if we develope according to the positive powers of ϱ, may be written $w^3=2+x+3.\overline{1+x+x^2}|^{\frac{1}{3}}w+\underset{1}{R}$, $\underset{1}{R}$ representing a series of terms of which every term is concerned with positive powers of ϱ, we have, by subtraction, $w^3-u^3=3(1+x+x^2)^{\frac{1}{3}}.(w-u)+\underset{1}{R}$; let $w=u+\varrho'$, and it will become, after transposition, $(3.u^2-3.\overline{1+x+x^2}|^{\frac{1}{3}})\varrho'+3u\varrho'^2+\varrho'^3=\underset{1}{R}$; hence ϱ' may be expressed by reversion, by a series of which every term is concerned with a positive power of $\underset{1}{R}$; and therefore, by restitution, since u and x are independent of ϱ, ϱ' may be expressed by a series (in which u and x are found) of which every term is concerned with ϱ; and consequently the sum of the series $1-\frac{2.5x^3}{3.6.9}-\frac{2.5.8.11.14}{3.6.9.12.15.18}x^6-\&c.=\frac{\overline{1-x}|^{\frac{1}{3}}+u}{3}+\varrho'-R=$ barely $\frac{\overline{1-x}|^{\frac{1}{3}}+u}{3}$; because all terms affected with ϱ vanish, from what has been already said; that is, ϱ' is $=R$, and u is one of the roots of the equation $u^3=2+x+3.\overline{1+x+x^2}|^{\frac{1}{3}}u$, and u may be found by the trigonometrical mode of extracting the root of cubic equations; thus, put $A=$ arc, whose cosine is $\frac{2+x}{2\sqrt[2]{1+x+x^2}}$, or whose tangent is $\frac{x\sqrt[2]{3}}{2+x}$, then will $u=2.\overline{1+x+x^2}|^{\frac{1}{6}}$cos.of $\left(\frac{A}{3}\right)$, and the sum of the series $1-\frac{2.5x^3}{3.6.9}-\frac{2.5.8.11.14}{3.6.9.12.15.18}x^6-\&c.$ is $=\frac{\overline{1-x}|^{\frac{1}{3}}+u}{3}$; but it is to be observed, that as u has three values which will satisfy the equation $u^3=2+x+3.\overline{1+x+x^2}|^{\frac{1}{3}}.u$, so if we wish to have the

value of the series $1-\frac{2.5x^3}{3.6.9}$, &c., we must not take for A any arc satisfying the equation A= arc whose tangent is $\frac{x\sqrt{3}}{2+x}$, but the arc to be taken for A, if positive, is that which is less than 90°, or, if we please, the same increased by any multiple of three circumferences; in fact, taking A in that manner, it will be $=\frac{x\sqrt[2]{3}}{2+x}-\overline{\frac{x\sqrt[2]{3}}{2+x}}\Big|^3.\frac{1}{3}+$ &c. $=\frac{x\sqrt{3}}{2}\times\overline{1-\frac{x}{2}+0}$, &c., and therefore the cosine of $\frac{1}{3}A=1-\frac{x^2}{24}+\frac{x^3}{24}$, &c., and $\overline{1+x+x^2}|^{\frac{1}{6}}$ $=1+\frac{x}{6}+\frac{7x^2}{72}-\frac{125x^3}{72\times 18}$, &c.; this multiplied by 2cos. of $\frac{1}{3}A$, or its equal $2-\frac{x^2}{12}+\frac{x^3}{12}$, &c. will give $2+\frac{x}{3}+\frac{2x^2}{18}-\frac{20x^3}{18\times 9}$, &c.: to this add $\overline{1-x}|^{\frac{1}{3}}$, or its equal $1-\frac{1}{3}x-\frac{2x^2}{18}-\frac{10x^3}{3.6.9}$, &c., and $\frac{1}{3}$ of the sum will give the very series proposed, namely, $1-\frac{2.5x^3}{3.6.9}-$ &c.

5. The two following are theorems of very extensive use in the application of imaginary quantities to analysis; namely, that if z be an arc of a circle whose radius is unity, x the sine and y the cosine of that arc, then is $x=\frac{\varepsilon^{z\sqrt{-1}}-\varepsilon^{-z\sqrt{-1}}}{2\sqrt{-1}}$, and $y=\frac{\varepsilon^{z\sqrt{-1}}+\varepsilon^{-z\sqrt{-1}}}{2}$, ε representing the number whose hyperbolical logarithm is unity. Now, instead of these forms, I shall propose for our consideration the two expressions, $\frac{\varepsilon^{z\sqrt{\varrho-1}}-\varepsilon^{-z\sqrt{\varrho-1}}}{2\sqrt{\varrho-1}}$, and $\frac{\varepsilon^{z\sqrt{\varrho-1}}+\varepsilon^{-z\sqrt{\varrho-1}}}{2}$. Put $x=\frac{\varepsilon^{z\sqrt{\varrho-1}}-\varepsilon^{-z\sqrt{\varrho-1}}}{2\sqrt{\varrho-1}}+R$, and $y=\frac{\varepsilon^{z\sqrt{\varrho-1}}+\varepsilon^{-z\sqrt{\varrho-1}}}{2}+R'$, x and y being the sine and cosine of z; then be-

cause $\varepsilon^{z\sqrt{\varrho-1}} = 1 + \sqrt{\varrho-1}z + \frac{\overline{\varrho-1}|^{\frac{2}{2}}}{2}z^2 + \frac{\overline{\varrho-1}|^{\frac{3}{2}}.z^3}{2.3} + \frac{\overline{\varrho-1}|^{\frac{4}{2}}.z^4}{2.3.4} +$ &c., $\varepsilon^{-z\sqrt{\varrho-1}} = 1 - \sqrt{\varrho-1}.z + \frac{\overline{\varrho-1}|^{\frac{2}{2}}.z^2}{2} - \frac{\overline{\varrho-1}|^{\frac{3}{2}}}{2.3}z^3 + \frac{\overline{\varrho-1}|^{\frac{4}{2}}z^4}{2.3.4} +$ &c., $x = z - \frac{z^3}{2.3} + \frac{z^5}{2.3.4.5} - \frac{z^7}{2.3.4.5.6.7} +$ &c., and $y = 1 - \frac{z^2}{2.3} + \frac{z^5}{2.3.4.5} -$ &c., we have, by substitution, $z - \frac{z^3}{2.3} + \frac{z^5}{2.3.4.5} -$ &c. $= z + \frac{\varrho-1}{2.3}z^3 + \frac{\overline{\varrho-1}|^2 z^5}{2.3.4.6} +$ &c. $+ R$; therefore $R = -\frac{\varrho z^3}{6} \times \left(1 + \frac{\varrho-2}{4.5}z^2 + \frac{\varrho^2-3\varrho+3}{4.5.6.7}z^4 + \frac{\varrho^3-4\varrho^2+6\varrho-4}{4.5.6.7.8.9}.z^6 + \&c.\right)$, and is consequently a series of terms of which every one is multiplied into a positive power of ϱ; and in the same manner we find $R' = -\frac{\varrho z^2}{1.2} - \frac{\varrho^2-3\varrho}{1.2.3.4}z^4 - \frac{\varrho^3-3\varrho^2+3\varrho}{1.2.3.4.5.6}z^6 +$ &c., and is a series of terms, every one of which is multiplied by a positive power of ϱ; hence, if in any investigation we use these values for x and y (of which every part is real), it is evident that, after proper reduction, ϱ, R, and R′, must vanish from the equation, and that, in consequence, nothing would have been wrong by suffering them to vanish at the commencement of the analysis, or had they been put equal to 0, or, in fact, any value most serviceable for our purpose.

Moreover, as $\frac{\varepsilon^{z\sqrt{\varrho-1}} + \varepsilon^{-z\sqrt{\varrho-1}}}{2} = y - R'$, and $\frac{\varepsilon^{z\sqrt{\varrho-1}} - \varepsilon^{-z\sqrt{\varrho-1}}}{2\sqrt{\varrho-1}} = x - R$, we have $\varepsilon^{2z\sqrt{\varrho-1}} + 2 + \varepsilon^{-2z\sqrt{\varrho-1}} = 4.\overline{y-R'}|^2$, and also $\varepsilon^{2z\sqrt{\varrho-1}} - 2 + \varepsilon^{-2z\sqrt{\varrho-1}} = 4.\overline{\varrho-1}|.\overline{x-R}|^2$, and therefore from the two $\overline{y-R'}|^2 - \overline{\varrho-1}.\overline{x-R}|^2 = 1$; and therefore $R' = y - \sqrt{\overline{\varrho-1}.\overline{x-R}|^2+1}$. We also get, from the equations $\frac{\varepsilon^{z\sqrt{\varrho-1}} + \varepsilon^{-z\sqrt{\varrho-1}}}{2} = y - R'$, and $\frac{\varepsilon^{z\sqrt{\varrho-1}} - \varepsilon^{-z\sqrt{\varrho-1}}}{2} = \sqrt{\varrho-1}.\overline{x-R}$, $\varepsilon^{z\sqrt{\varrho-1}}$

$=y-R'+\sqrt{\varrho-1}\,.\,\overline{x-R}$, and $\varepsilon^{-z\sqrt{\varrho-1}}=y-R'-\sqrt{\varrho-1}\,.\,\overline{x-R}$; these, if ϱ were taken equal to 0, would give the known imaginary formulæ $\varepsilon^{z\sqrt{-1}}=y+\sqrt{-1}\,.\,x$, and $\varepsilon^{-z\sqrt{-1}}=y-\sqrt{-1}\,.\,x$. If $z=\frac{1}{4}$ of the circumference of a circle whose radius is unity, then will $y=0$ and $x=1$, and therefore $\varepsilon^{z\sqrt{-1}}=\sqrt{-1}$, and therefore $z=$ hyp. log. of $\frac{\sqrt{-1}}{\sqrt{-1}}$, J. Bernouilli's Theorem. Also, from $\varepsilon^{z\sqrt{-1}}=\sqrt{-1}$ we get $\varepsilon^{z\sqrt{-1}}$ raised to $\sqrt{-1}$ power, that is, $\varepsilon^{-z}=\overline{\sqrt{-1}}|^{\sqrt{-1}}$, Euler's Theorem. But instead of these theorems, we may use theorems of real arithmetical quantity; for, in the case of z being a quarter of the circumference of the circle whose radius is unity, we have, from the general equations $\varepsilon^{z\sqrt{\varrho-1}}=y-R'+\sqrt{\varrho-1}\,.\,\overline{x-R}$, simply $\varepsilon^{z\sqrt{\varrho-1}}=\sqrt{\varrho-1}\,.\,\overline{1-R}-R'$, and therefore $z=\frac{\text{hyp. log. of }(\sqrt{\varrho-1}\,.\,\overline{1-R}-R')}{\sqrt{\varrho-1}}$, R and R′ being as above; also, from $\varepsilon^{z\sqrt{\varrho-1}}=\sqrt{\varrho-1}\,.\,\overline{1-R}-R'$ we get $\overline{\varepsilon^{z\sqrt{\varrho-1}}}|^{\sqrt{\varrho-1}}$, that is, $\varepsilon^{z\,.\,\varrho-1}=\overline{\sqrt{\varrho-1}\,.\,\overline{1-R}-R'}|^{\sqrt{\varrho-1}}$, and therefore $\varepsilon^{-z}=\frac{\overline{\sqrt{\varrho-1}\,.\,\overline{1-R}-R'}|^{\sqrt{\varrho-1}}}{\varepsilon^{\varrho z}}$, whatever ϱ may be (at least whilst the expression is real); and therefore if ϱ were taken equal to 0, we again have the imaginary equation $\varepsilon^{-z}=\overline{\sqrt{-1}}|^{\sqrt{-1}}$.

Moreover, if X be the sine of nz, and Y the cosine, we shall have from the theorems $x=\frac{\varepsilon^{z\sqrt{\varrho-1}}-\varepsilon^{-z\sqrt{\varrho-1}}}{2\sqrt{\varrho-1}}-\frac{\varrho z^3}{6}\left(1+\frac{\varrho-2}{4.5}\,.\,z^2+\frac{\varrho^2-3\varrho+3}{4.5.6.7}z^4,\ \&c.\right)$ and

$y=\dfrac{\varepsilon^{z\sqrt{\varrho-1}}-\varepsilon^{-z\sqrt{\varrho-1}}}{2\sqrt{\varrho-1}}-\dfrac{\varrho z^2}{2}\times\left(1+\dfrac{\varrho-2}{2.3}z^2+\dfrac{\varrho^2-3\varrho+3}{2.3.4.5}.z^4,\ \&c.\right)$; $X=$ $\dfrac{\varepsilon^{nz\sqrt{\varrho-1}}-\varepsilon^{-nz\sqrt{\varrho-1}}}{2\sqrt{\varrho-1}}-\dfrac{\varrho n^2z^3}{6}\times\left(1+\dfrac{\varrho-2}{4.5}.n^2z^2+\&c.\right)$, and $Y=$ $\dfrac{\varepsilon^{nz\sqrt{\varrho-1}}+\varepsilon^{-nz\sqrt{\varrho-1}}}{2}-\dfrac{\varrho n^2z^2}{2}\times\left(1+\dfrac{\varrho-2}{2.3}n^2z^2+\ \&c.\right)$; or, if we put $\underset{n}{R}=-$ $\varrho.\dfrac{n^2z^3}{6}\times\left(1+\dfrac{\varrho-2}{4.5}n^2z^2+\ \&c.\right)$, and $\underset{n}{R}'=-\dfrac{\varrho n^2}{2}\times\left(1+\dfrac{\varrho-2}{4.5}.n^2z^2+\ \&c.\right)$, we have X, or the ſine of $nz=\dfrac{\varepsilon^{nz\sqrt{\varrho-1}}-\varepsilon^{-nz\sqrt{\varrho-1}}}{2\sqrt{\varrho-1}}+\underset{n}{R}$, and Y or cos. of $nz=$ $\dfrac{\varepsilon^{nz\sqrt{\varrho-1}}+\varepsilon^{-nz\sqrt{\varrho-1}}}{2}+\underset{n}{R}'$; also, $\varepsilon^{nz\sqrt{\varrho-1}}=$ cos. of $\overline{nz}+\sqrt{\varrho-1}$. sine of $\overline{nz}-$ $\underset{n}{R}'-\sqrt{\varrho-1}.\underset{n}{R}$; but $\varepsilon^{z\sqrt{\varrho-1}}=$ cos. of $z+\sqrt{\varrho-1}$ sine of $z-R'-\sqrt{\varrho-1}\underset{n}{R}$; therefore cos. of $nz+\sqrt{\varrho-1}$ sine of $nz-\underset{n}{R}'-\sqrt{\varrho-1}\underset{n}{R}=($cos. of $z+\sqrt{\varrho-1}$. sine of $z-R'-\sqrt{\varrho-1}R)^n=\overline{\text{cos. of } z+\sqrt{\varrho-1}\text{ . sine of } z}|^n+Q$, Q representing the whole developement of $\overline{\text{cos. of } z+\sqrt{\varrho-1}\text{ . sine of } z-R'-\sqrt{\varrho-1}.R}|$, excepting $\overline{\text{cos. of } z+\sqrt{\varrho-1}\text{ . sine of } z}|^n$, and consequently every term of the series, whether finite or infinite, of which Q is made up will be either multiplied by R' or R, and consequently, by restitution, each term will be multiplied by ϱ; consequently, if $\underset{n}{R}'+\sqrt{\varrho-1}.\underset{n}{R}+Q$ be put $=Q'$, Q' will likewise be a series of terms of which every part is multiplied by ϱ, and we shall have $\overline{\text{cos. of } z+\sqrt{\varrho-1}\text{ . sine of } z}|^n=$ cos. of $nz+$ $\sqrt{\varrho-1}$. sine of $nz-Q'$; and in a similar manner may P' be found, being a series of terms of which every term is multiplied by ϱ, such that $\overline{\text{cos. of } z-\sqrt{\varrho-1}\text{ . sine of } z}|^n=$ cos. of $nz-\sqrt{\varrho-1}$. sine of $nz-P'$.

These two last theorems, if ϱ were taken equal to 0, would be the well known theorems $\overline{\text{cos.of } z+\sqrt{-1}\text{ . sine of } z}|^n=\text{cos. of } nz+\sqrt{-1}\text{ sine of } nz$, and $\overline{\text{cos.of } z-\sqrt{-1}\text{ . sine of } z}|^n=\text{cos.of } nz-\sqrt{-1}\text{ . sine of } nz$, under the imaginary form; but whatever ϱ be taken they must revert to the same thing, since, ϱ being arbitrary, it must vanish, as all along insisted on, after developement; and therefore, by taking it previously equal to 0, or any other convenient value, we can produce nothing wrong.

It is evident from the above real formulæ, by addition and subtraction, &c. we obtain cos.of $nz=$

$$\frac{\overline{\text{cos.of } z+\sqrt{\varrho-1}\text{ . sine of } z}|^n+\overline{\text{cos.of } z-\sqrt{\varrho-1}\text{ . sine of } z}|^n+Q'+P'}{2}$$ and sine of $nz=\dfrac{\overline{\text{cos.of } z+\sqrt{\varrho-1}\text{ . sine of } z}|^n-\overline{\text{cos.of } z-\sqrt{\varrho-1}\text{ . sine of } z}|^n+Q'-P'}{2\sqrt{\varrho-1}}$;

and if these be developed, and every term which is multiplied by ϱ be omitted, by reason of the necessity already proved of their destroying each other, we get the same as would be obtained by using the imaginary formulæ cos.of $nz=$

$$\frac{\overline{\text{cos. of } z+\sqrt{-1}\text{ . sine of } z}|^n+\overline{\text{cos.of } z-\sqrt{-1}\text{ . sine of } z}|^n}{2}$$ and sine of $nz=$

$$\frac{\overline{\text{cos.of } z+\sqrt{-1}\text{ . sine of } z}|^n-\overline{\text{cos.of } z-\sqrt{-1}\text{ . sine of } z}|^n}{2\sqrt{-1}},$$ that is, we obtain $\text{cos.of } nz=\overline{\text{cos. of } z}|^n-n\,.\,\frac{n-1}{2}\,.\,\overline{\text{sine of } z}|^2\,.\,\overline{\text{cos.of } z}|^{n-2}+n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}\,.\,\frac{n-3}{4}\,.\,\overline{\text{sine of } z}|^4\,.\,\overline{\text{cos.of } z}|^{n-4}-\&c.$ and the sine of $nz=n\overline{\text{cos.of } z}|^{n-1}\,.\,\text{sine of } z-n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}\,.\,\overline{\text{cos.of } z}|^{n-3}\,.\,\overline{\text{sine of } z}|^3+\&c.$

6. Imaginary quantities are used very advantageously in the resolution of certain fluxional equations; for instance, in equations of

the form $y+\alpha\frac{\dot{y}}{\dot{x}}+\beta\frac{\ddot{y}}{\dot{x}^2}+\gamma\frac{\dddot{y}}{\dot{x}^3}$&c.$=0$; and we are conducted to imaginary quantities in virtue of the property of the exponential ε^{mx}, where ε represents the number whose hyperbolical logarithm is unity, and m a constant quantity. The property to which I allude is, that the first fluxion of this exponential is $m\,.\,\varepsilon^{mx}\dot{x}$, the second fluxion $m^2\varepsilon^{mx}\dot{x}^2$, the third fluxion $m^3\varepsilon^{mx}\dot{x}^3$, &c.; consequently, if in the equation above we assume $y=A\varepsilon^{mx}$, A being constant, it will become $A\varepsilon^{mx}\times(1+\alpha m+\beta m^2+\gamma m^3+\&c.)=0$, or simply $1+\alpha m+\beta m^2+\gamma m^3+$,&c.$=0$, m being constant; and if α, β, γ*, &c. be constant quantities, m may be found to satisfy this equation either by a real or an imaginary quantity. There have been very instructive particulars drawn from the present problem, well worthy the attention of the mathematician; namely, the manner of proceeding when the equation $1+\alpha m+\beta m^2+$,&c.$=0$ has all real roots differing from each other, when some of the roots are equal to each other, and when some or all are imaginary.

I shall propose, as belonging to our present object, To find the complete fluent answering the equation $\overset{\cdot\cdot\cdot\cdot\cdot}{y}+y\dot{x}^5=0$. Take $y=A\varepsilon^{mx}$, A standing for a constant quantity, ε being the number whose hyperbolic logarithm is unity; therefore $\dot{y}=m\dot{x}A\varepsilon^{mx}$, $\ddot{y}=m^2\dot{x}^2A\varepsilon^{mx}$, &c., $\overset{\cdot\cdot\cdot\cdot\cdot}{y}=m^5\dot{x}^5A\,.\,\varepsilon^{mx}$: this being substituted in the proposed equation, we obtain $m^5+1=0$, $m=-1$ will answer this requisite, and we shall in consequence have $y=A\varepsilon^{-x}$ for a solution. But this contains only one arbitrary constant, whereas, from the theory of fluxional equations for a complete solution, it ought to contain five arbitrary constants; and the analyst acquainted with the doctrine of imaginary quantities seeks assistance from the imaginary roots of the equation $m^5+1=0$. Now, from what we have already seen, we have cos. of $nz+\sqrt{-1}\,.$ sine of $nz=\overline{\text{cos. of } z+\sqrt{-1}\,.\text{ sine of } z}|^n$; consequently if we take $n=5$ and z so that the cosine of nz is $=-1$, and therefore sine of $nz=0$, we shall have $\overline{\text{cos. of } z+\sqrt{-1}\,.\text{ sine of } z}|=-1$;

* If α, β, γ, &c. were not constant, $1+\alpha m+\beta m^2=0$ would only be possible in *very* particular cases if m were constant.

but $m^5=-1$, therefore $m=$ cos.of $z+\sqrt{-1}$. sine of z. Now, to fulfil the conditions of cos.of $nz=-1$, that is, cos.of $5z=-1$, we may take $5z=180°$, $180°+360°$, $180°+360°\times 2$, $180°+360°\times 3$, $180°+360°\times 4$, &c.; but it is not necessary to proceed, as these will give all the values that m can have, and we should have the same values of m by proceeding as by using the five values of $5z$ above; hence, for z we are to take the values $36°$, $36°+72°$, $36°+144°$, $36°+216°$, and $36°+288°$; and consequently we have for m any of the five expressions,

cos.of $36°+\sqrt{-1}$. sine of $36°$

cos.of $108°+\sqrt{-1}$ sine of $108°$, that is $-$cos.of $72°+\sqrt{-1}$. sine of $72°$,

cos.of $180°+\sqrt{-1}$ sine of $180°=-1$,

cos.of $252°+\sqrt{-1}$ sine of $252°=-$cos.of $72°-\sqrt{-1}$. sine of $72°$, and

cos.of $324°+\sqrt{-1}$ sine of $324°=$ cos.of $36°-\sqrt{-1}$. sine of $36°$;

and consequently if a, a' be put for the sine and cosine of $36°$, and b, b' for the sine and cosine of $76°$, we have for m either of the expressions following, $a'+a\sqrt{-1}$, $-b'+b\sqrt{-1}$, 1, $a'-a\sqrt{-1}$ and $-b'-b\sqrt{-1}$; and we may take for the fluent, $y=A\,.\,\varepsilon^{a'x+ax\sqrt{-1}}+B\,.\,\varepsilon^{a'x-ax\sqrt{-1}}+C\,.\,\varepsilon^{-b'x+bx\sqrt{-1}}+D\,.\,\varepsilon^{-b'x-bx\sqrt{-1}}+E\varepsilon^{-x}$; A, B, C, D, and E being five arbitrary constants, and we may write it thus: $y=\varepsilon^{a'x}\times\left(A\varepsilon^{ax\sqrt{-1}}+B\,.\,\varepsilon^{-ax\sqrt{-1}}\right)+\varepsilon^{-b'x}\times\left(C\,.\,\varepsilon^{bx\sqrt{-1}}+B\varepsilon^{-bx\sqrt{-1}}\right)+E\,.\,\varepsilon^{-x}$; but from what we have seen, this may likewise be written $y=\varepsilon^{a'x}($A(cos.of $ax+\sqrt{-1}$ sine of $ax)+$B cos.of $(ax-\sqrt{-1}$ sine of $ax))+\varepsilon^{-bx'}\times($C(cos.of $bx+\sqrt{-1}$ sine of $bx)+$D(cos.of $bx-\sqrt{-1}$ sine of $bx))+E\,.\,\varepsilon^{-x}$; and, putting $A+B=A'$, $\overline{A-B}\sqrt{-1}=B'$, $C+D=C'$, and $\overline{C-D}\sqrt{-1}=D'$, we have $y=\varepsilon^{a'x}\times(A'$cos.of $ar+B'$ sine of $ax)+\varepsilon^{-b'x}\times(C'$cos.of $bx+D'$sine of $bx)+E\,.\,\varepsilon^{-x}$; A′, B′, C′, D′, and E being the five arbitrary constants. And to show how this may be imitated without introducing imaginary quantities, z being as before, that is, so that cos.of $5z=-1$, take $m=$ cos.of $z+\sqrt{\varepsilon-1}$. sine of $5z$, and $y=A\varepsilon^{mx}+K$, A being constant and K variable; consequently, as z is constant, we have $\frac{\overset{\cdot\cdot\cdot\cdot\cdot}{y}}{\dot{x}^5}=Am^5\,.\,\varepsilon^{mx}+\frac{\overset{\cdot\cdot\cdot\cdot\cdot}{K}}{\dot{x}^5}$, and

the equation to be resolved is $Am^5\varepsilon^{mx}+A\varepsilon^{mx}+\frac{\overset{.....}{K}}{\dot{x}^5}+K=0$; but $m^5=$ $\overline{\text{cos. of } z+\sqrt{\varrho-1}\,.\,\text{sine of } z}|^5$ $=$ (by Art. 5) cos. of $5z+\sqrt{\varrho-1}$. sine of $5z$ $-Q'$; Q' being put for a series, of which every term is multiplied by ϱ; and consequently, because cos. of $5z=-1$, and therefore sine of $z=0$, we have $m^5=-1-Q'$, and consequently the equation to be resolved is reduced to $\frac{\overset{.....}{K}}{\dot{x}^5}+K=Q'$, and therefore $K=Q'-\frac{\overset{.....}{K}}{\dot{x}^5}=Q'-$

$$\frac{\left(Q'-\frac{\overset{.....}{K}}{\dot{x}^5}\right)^{(.5)}}{\dot{x}|^5}=Q'-\frac{Q'^{(.5)}}{\dot{x}^5}+\frac{K^{(.10)}}{\dot{x}|^{10}}=^{*}Q'-\frac{Q'^{(.5)}}{\dot{x}|^5}+\frac{Q'^{(.10)}}{\dot{x}|^{10}}-\frac{Q'^{(.15)}}{\dot{x}|^{15}}\ \text{\&c.};$$ but

every term of Q' being multiplied by the arbitrary constant ϱ, every term of all the fluxions thereof will likewise be multiplied by ϱ, as ϱ is not concerned in Q' as an exponent of a power, nor as a transcendental; consequently K is likewise a series of terms of which every part is multiplied by ϱ: hence $y=A\varepsilon^{(\text{cos. of } z+\sqrt{\varrho-1}\,.\,\text{sine of } z)}+K$, z being such that the cos. of $5z=-1$; and therefore, putting as above a and a' for the sine and cosine of 36°, b and b' for the sine and cosine of 72°, we may take $y=A\varepsilon^{(a'+a\sqrt{\varrho-1})x}+B\,.\,\varepsilon^{(a'-a\sqrt{\varrho-1})x}+C\,.\,\varepsilon^{(b'+b\sqrt{\varrho-1})x}+D\varepsilon^{(b'-b\sqrt{\varrho-1})x}+E\,.\,\varepsilon^{-x}+$ $\underset{,}{K}$; $\underset{,}{K}$ being a series of terms of which every one is multiplied by ϱ. And this may be written $y=\varepsilon^{a'x}\times(A\varepsilon^{a\sqrt{\varrho-1}.x}+B\varepsilon^{-a\sqrt{\varrho-1}.x})+\varepsilon^{b'x}\times(C\varepsilon^{b\sqrt{\varrho-1}}$ $+D\varepsilon^{-b\sqrt{\varrho-1}})+E\,.\,\varepsilon^{-x}+\underset{,}{K}$; but from above we have, whatever ϱ is, $\varepsilon^{a\sqrt{\varrho-1}.x}=\text{cos of } \overline{ax-M'}+\sqrt{\varrho-1}\,.\,\overline{\text{sine of } \overline{ax-M}}$, $\varepsilon^{-a\sqrt{\varrho-1}.x}=\text{cos of } \overline{ax-M'}-\sqrt{\varrho-1}\,.\,\text{sine of } \overline{ax-M}$, $\varepsilon^{b\sqrt{\varrho-1}.x}=\text{cos. of } \overline{bx-N'}+\sqrt{\varrho-1}\,.\,\overline{\text{sine of } \overline{bx-N}}$ and $\varepsilon^{-b\sqrt{\varrho-1}.x}=\text{cos. of } \overline{bx-N'}+\sqrt{\varrho-1}\ \overline{\text{sine of } bx-N}$, M, M', N, N'

* $Q'^{(.5)}$, $Q'^{(.10)}$, &c. stand for the fifth, tenth, &c. fluxions of Q'.

being quantities of which every term is multiplied by ϱ; these are found, by proper substitution, in the expressions above given, namely, $\varepsilon^{x\sqrt{\varrho-1}}=y-\text{R}'+\sqrt{\varrho-1}\,.\,\overline{x-\text{R}}$ and $\varepsilon^{-x\sqrt{\varrho-1}}=y-\text{R}'-\sqrt{\varrho-1}\,.\,\overline{x-\text{R}}$: hence, by substitution, and putting $\text{A}+\text{B}=\text{A}'$, $\overline{\text{A}-\text{B}}\sqrt{\varrho-1}=\text{B}'$, $\text{C}+\text{D}=\text{C}'$, and $\overline{\text{C}-\text{D}}\sqrt{\varrho-1}=\text{D}'$, and considering A′, B′, C′, D′, and E arbitrary constants, independent of the arbitrary constant ϱ, we have $y=\varepsilon^{a'x}\times$ (A′ .cos.of $ax+\text{B}'$ sine of $ax)+\varepsilon^{-b'x}\times$ (C′cos.of $bx+\text{D}'$sine of $bx)+\text{E}\,.\,\varepsilon^{-x}$, the same as before, because the terms M′, M, N′, N and K_1 being each a series of terms multiplied by ϱ, which is independent of x and y, will, from what has been repeatedly urged, destroy each other.

In finding the integrals of equations of finite increments, the imaginary quantity very frequently occurs, as well in the necessary integral as in the correction. All the steps may be imitated by similar means to those I have used above, and the imaginary quantity avoided.

Imaginary quantities have been used, in the doctrine of series, for the determination of the sums of trigonometrical series. See Landen's Mathematical Memoirs.

I have shown, in the Philosophical Transactions, in a Paper read before the Royal Society, Feb. 13, 1806, how imaginary quantities may be avoided in those speculations of Landen, by the application of a method of differences, and a method will easily follow from this Tract for doing the same by imitating Landen's steps, and adding what our method points out.

I think it not improper to remark, that, whenever we indicate an equality between quantities which are in themselves incomparable, the meaning must be not a real equality, but a conditional equality arising from the condition of certain changes taking place.

Mathematics, in its present improved state, abounds with indications of such conditional equality, the equation $\Delta^n=\left(\varepsilon^{\frac{duh}{dx}}-1\right)^n$ signifying, ac-

cording to the French notation, the increment taken n successive times of u, is $=\left(\varepsilon^{\frac{\dot{u}}{\dot{x}}\cdot x}-1\right)^n$ is only true after certain changes have taken place. And in this way do I understand the sign $\doteq$, when found in certain imaginary equations; and the conditions are, that in operating with $\sqrt{-1}$, we are to operate as we would with $\sqrt{\varrho-1}$, and then afterwards expunge ϱ; or, shorter, only to suppose ϱ to be there without inserting it. It might be thought more consistent with reason to use a different sign when conditional equality is meant to be expressed than when real equality is to be inferred, and no doubt it would where we should otherwise be able to draw wrong inferences.

7. The Reader, during his perusal of this Tract, had an opportunity of observing that the introduction of the arbitrary quantity in the different expressions which consist of imaginary parts, independent of that introduction, had the effect of giving reality to separate parts of the real results, and that the results did not at all alter by changing the value of the arbitrary quantities introduced; and, consistent with the definition of *porisms*, I think I may venture to call expressions consisting of such arbitrary quantities, *porismatic* expressions, and likewise to use the term to *porismatise*, an expression to signify the act of introducing one arbitrary quantity or more into a formula, such that the expression shall be constantly of the same value, whether or not the arbitrary quantity or quantities be varied. This is an indeterminate problem, which, generally taken, appears to me to offer a *new* and extensive field for speculation. Let us now consider the indeterminate problem, To porismatise or to complete the expression $\sqrt[2]{a+b\,.\sqrt[2]{-1}}+\sqrt[2]{a-b\sqrt[2]{-1}}$. Put it equal to $\sqrt[2]{a+m+b\sqrt[2]{\varrho-1}}+\sqrt[2]{a+m-b\sqrt[2]{\varrho-1}}$, and square both sides of the equation, and we have, after reduction, $2a+2.\sqrt[2]{a^2+b^2}=2a+2m+2.\sqrt[2]{\overline{a+m}|^2-b^2.\overline{\varrho-1}}$; and consequently $\sqrt[2]{a^2+b^2}-m\doteq\sqrt[2]{a^2+2am+m^2-b^2\,.\,\overline{\varrho-1}}$; square both sides, and we get, after

reduction, $-2m\sqrt[2]{a^2+b^2}=2am-b^2.\varrho$; and consequently $m=\frac{b^2.\varrho}{2a+2\sqrt[2]{a^2+b^2}}$, or its equal $\frac{-a-\sqrt[2]{a^2-b^2}}{2}.\varrho$; and consequently the formula, filled up, completed, or porismatised, may be $\sqrt[2]{a+\frac{1}{2}\varrho.(-a-\sqrt[2]{a^2-b^2})+b\sqrt[2]{\varrho-1}}+\sqrt[2]{a+\frac{1}{2}\varrho.(-a-\sqrt[2]{a^2-b^2})-b\sqrt[2]{\varrho-1}}$, an expression containing an arbitrary quantity ϱ; which expression, if a and b are given quantities, is itself given, though ϱ may vary. If ϱ be taken equal to 0, this expression will become the proposed expression $\sqrt[2]{a+b\sqrt[2]{-1}}+\sqrt[2]{a-b\sqrt[2]{-1}}$.

The above fact may be enunciated as a porism in the following manner:—If a and b be given, A is likewise given, such that, whatever ϱ may be, $\sqrt[2]{a+A\varrho+b\sqrt[2]{\varrho-1}}+\sqrt[2]{a+A\varrho-b.\sqrt[2]{\varrho-1}}$ will be given: and the investigation of it may be as follows, independent of imaginary quantities.

Suppose it true; then, because the expression is always given, whatever ϱ may be, therefore, taking $\varrho=1$, we find it to be $=2\sqrt{a+A}$; that is, if the position be true, we have $\sqrt[2]{a+A\varrho+b.\sqrt[2]{\varrho-1}}+\sqrt{a+A\varrho-b.\sqrt[2]{\varrho-1}}=2\sqrt{a+A}$; square both sides, and we have $2a+2A\varrho+\sqrt[2]{\overline{a+A\varrho}^2-b^2.\overline{\varrho-1}}.=4a+4A$; consequently $\sqrt{\overline{a+A\varrho}^2-b^2.\overline{\varrho-1}}=a+\overline{2-\varrho}A$; and, squaring $a^2+2aA\varrho+A^2\varrho^2-b^2\varrho+b^2=a^2+4aA-2aA\varrho+4A^2-4\varrho A^2+\varrho^2A^2$; consequently $b^2.\overline{\varrho-1}-4aA.\overline{\varrho-1}-4A^2.\overline{\varrho-1}=0$; and therefore, dividing by $\overline{\varrho-1}$, $b^2-4aA-4A^2=0$, and consequently $A=\frac{-a\mp\sqrt{a^2+b^2}}{2}$. Q.E.D.

The same may be enunciated as a porism concerning lines, as follows:—

Given three right lines G, A, *and* B; *there is given a fourth right line* R, *such that, if* r, q, p, *be any right lines at pleasure, constituting respectively the base, perpendicular, and hypothenuse of a right angled triangle, and* X *be a fourth proportional to the square on* q, *the square on* p, *and the right line* R; *and* Z *a fourth proportional to the right*

lines q, r, *and* B; *then, if* P *be a mean proportional between* G *and the sum of the right lines* A, X, *and* Z, *and* Q *a mean proportional between* G *and the excess of* A *and* X *above* Z, *the sum of* P *and* Q *shall be given.*

The Reader is requested to pay particular attention to the geometrical investigation of this porism which I am now about to offer, as I am purposely leading him through a route in which imaginary quantities may be passed over by the geometrical traveller, without, perhaps, his being aware that he is treading on a ground which by many is considered unworthy to be trodden by the geometrician's foot, and which is even by some believed not to be possibly found in his way. Suppose it true; then, since the sum of the right lines P and Q are given, the square on this sum is likewise given; that is, the sum of the spaces which are the square on P, the square on Q, and the double rectangle contained under P and Q is given; but the square on P is equal to the sum of the rectangles contained under G and A, G and X, and G and Z; and the square on Q is equal to the excess of the rectangles under G and A, and G and X above the rectangle under G and Z; consequently the sum of the square on P and square on Q is equal to the sum of double the rectangles contained under G and A and under G and X. Let L be a fourth proportional between G, P, and Q, and consequently, because the sum of the spaces which are the square on P, the square on Q, and double the rectangle contained under the right lines P and Q, is given, we have its equal, the sum of the spaces which are double the rectangles under G and A, under G and X, and under G and L, given; and consequently, because G and A are given, the sum of X and L will be given. Let M be the right line which is equal to the sum of the right lines X and L; then will M be given, and L will be equal to the excess of M above X; and consequently the square on L will be equal to the excess of the sum of the squares on M and on X above the double rectangle under M and X. But we have taken L, so that, as L is to P so is Q to G; and consequently the square on L is

to the square on P so is the square on Q to the square on G; but the square on P is equal to the rectangle under G and the right line which is equal to the sum of A, X, and Z; and the square on Q is equal to the rectangle under G and the right line which is equal to the excess of the sum of A and X above Z; consequently, from the proportion just mentioned, we see that the square on L is equal to the rectangle under two right lines, of which the one is equal to the sum of A, X, and Z, and the other the excess of A and X above Z; and consequently that the square on L is equal to the excess of the square on the sum of the lines A and X above the square on Z, that is, its equal the excess of the square on A, the square on X, and the double rectangle under A and X together above the square on Z. But the square on L was also above shown to be equal to the excess of the sum of the squares on M and on X above the double rectangle under M and X, and consequently, by comparison of these two values of the square on L, we find that the excess of the square on A, together with the double rectangle under A and X above the square on Z, is equal to the excess of the square on M above the double rectangle under M and X.

Furthermore, take W a fourth proportional to q, p, and B; consequently the square on q is to the square on p, so is the square on B to the square on W; and therefore the excess of the square on p above the square on q, that is, the square on r is to the square on q, so is the excess of the square on W above the square on B to the square on B;, but, by hypothesis, q is to r as B is to Z; consequently the square on r is to the square on q, so is the square on Z to the square on B; that is, from immediately above, as the excess of the square on W above the square on B to the square on B: whence the square on Z is equal to the excess of the square on W above the square on B; and therefore, because we have shown that the excess of the square on A, together with the double rectangle under A and X, above the square on Z is equal to the excess of the square on M above the double rectangle under M and X, it follows that the excess of the square on A, together with the

double rectangle under A and X and the square on B, above the square on W, is equal to the excess of the square on M above the double rectangle under M and X. But M, if the porism be true, has been proved to be given (though p be variable); consequently, if p be ever so small, or even evanescent, M will still be the same; but if p be evanescent, q not being so, W and X would both be evanescent; we should therefore have, from the general case, that the excess of the square on A, together with double the rectangle under A and X; and the square on B above the square on W is equal to the excess of the square on M above the double rectangle under M and X; likewise, that the square on A, together with the square on B, is equal to the square on M; and therefore, by subtraction, the excess of the square on W, above the double rectangle under A and X, is equal to the double rectangle under M and X, and consequently the square on W is equal to the double rectangle under the sum of the lines M and A, and the line X. But we have already said that the square on q is to the square on p as the square on B is to the square on W; and likewise, by the proposition, as R is to X; therefore the square on B is to the double rectangle under the sum of A and M, and X, as R is to X; and consequently, as twice the sum of A and M is to B, so is B to R. But M is given; consequently R is given. Q.E.D.

In the investigation of the truth of porisms, besides the consideration of the case proposed generally, it is commonly found of advantage to consider certain particular cases of the porisms, in which they may become more simple, being rendered so by certain terms vanishing in those cases: thus, in Simson's first Porism, a certain property is to be shown to be true in whatever manner the right line FEG is drawn through the point E, or, in other words, whatever be the angle HEF; and, besides the general case, Simson assumes a particular case, in which the angle HEF is nothing: the assumption of such cases of *ease* continually occur in Simson's *Treatise on Porisms*. The case of *ease* which I have used is, when p vanishes under the supposition of q not vanishing; but p

being the hypothenuse, and q one of the legs of a right angled triangle, the case is impossible or imaginary, and the lines P and Q would be also imaginary; but still I contend, that the investigation is perfectly geometrical; for, though we have assumed a case incompatible with the porism proposed, still, as that case is not all incompatible with the general equality which was under consideration, namely, that the excess of the square on A, together with the double rectangle under A and X and the square on B above the square on W, is equal to the excess of the square on M above the double rectangle under M and X, that case may be assumed as far as regards this equality; but this equality cannot take place throughout the whole limits of the proposition, unless it takes place generally; and therefore the assumed case being incompatible with the limitations of the proposition, can by no means affect the validity of the argument.

It is plain that the assumption of the imaginary case of *ease* may be avoided; and, had we taken the case of q evanescent instead of p, that object would have been accomplished.

To dwell long on the subject of porisms would be carrying our views from the main object of this Tract; still, as we have been led to consider that branch from the nature of our immediate object, the Reader might consider me leaving him too abruptly were I at this moment to quit the subject. I shall therefore remark, that it is not only quantities consisting of imaginary parts which may become the object of this, *I believe, new* speculation, as will be evident from what follows.

Problem.—Porismatise $a.b$.

Solution.—Put it $=\sqrt{a^2-\varrho^2}\sqrt{b^2+m^2}$; consequently $a^2.b^2=a^2b^2+a^2m^2-b^2\varrho^2-\varrho^2m^2$; therefore $m=\frac{b\varrho}{\sqrt{a^2-\varrho^2}}$, and we may state, in the form of a porism, that a and b being given, k will likewise be given; so that, if $m=\frac{k\varrho}{\sqrt{a^2-\varrho^2}}$, $\sqrt{a^2-\varrho^2}\sqrt{b^2+m^2}$ is given, whatever ϱ may be. And if we investigate this, we shall find $k=b$ and $\sqrt{a^2-\varrho^2}\sqrt{b^2-m^2}=ab$, whatever ϱ is. Again, take the same expression ab, and put it $=\overline{a-m}\times\overline{b+\varrho}=ab+$

$a\varrho - bm - \varrho m$; therefore $m = \frac{a\varrho}{b+\varrho}$, and we may state as a porism, a and b being given, k is given; so that, if $m = \frac{k\varrho}{b+\varrho}$, $\overline{a-m}\,.\,\overline{b+\varrho}$ will be given, whatever ϱ may be, and k will be found $=a$, and $\overline{a-m}\,.\,\overline{b+\varrho} = ab$.

Take $a + \sqrt{a^2+b^2}$. Put it $= a + m + \sqrt{\overline{m+a}|^2 - \varrho^2 + b^2}$; therefore $m^2 - 2m\sqrt{a^2+b^2} + a^2 + b^2 = m^2 + 2ma + a^2 - \varrho^2 + b^2$; therefore $2\,.\,(a + \sqrt{a^2+b^2})m = \varrho^2$, and we may state, that if a and b are given, r is given; so that if ϱ be a mean proportion between r and m, $a + m + \sqrt{\overline{m+a}|^2 - \varrho^2 + b^2}$ will be given, and by the investigation of the porism we shall find $r = 2\,.\,(a + \sqrt{a^2+b^2})$.

We might proceed now to a description of porisms containing two or more variables; but I prefer leaving that speculation for the present. The second Tract of these Series will, I think, contain some new subjects on imaginary quantities, in their application to geometry.

END OF BOOK I. AND TRACT I.

ERRATA.

Page vii, last word, for *producted* read *product*. Page 11, lines 4 and 9 from the bottom, for 2*cspr* read 2*csprq*. Page 22, line 7 from bottom, for $\overline{\varepsilon^{z\sqrt{\varrho-1}}}|^{\sqrt{\varrho-1}}$ read $\overline{\varepsilon^{z\sqrt{\varrho-1}}}|^{\sqrt{\varrho-1}}$. Page 23, line 7, for the last $\underset{n}{R}$ read R. Page 28, last line, for Δ^n read $\Delta^n u$.

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